basic description of the mean velocity profiles in self-similar free shear flows. However, it is an incomplete model with a very limited range of applicability.

10.2.2 Mixing-Length Model

In application to two-dimensional boundary layer flows, the mixing length $\ell_m(x, y)$ is specified as a function of position, and then the turbulent viscosity is obtained as

$$\nu_T = \ell_m^2 \left| \frac{\partial \langle U \rangle}{\partial y} \right|. \tag{10.19}$$

As shown in Section 7.1.7, in the log-law region, the appropriate specification of the mixing length is $\ell_m = \kappa y$, and then the turbulent viscosity is $\nu_T = u_\tau \kappa y$.

Several generalizations of Eq. (10.19) have been proposed to enable the application of the mixing-length hypothesis to all flows. Based on the mean rate-of-strain $\bar{S}_{ij}$ Smagorinsky (1963) proposed

$$\nu_T = \ell_m^2 (2\bar{S}_{ij}\bar{S}_{ij})^{\frac{1}{2}} = \ell_m^2 \mathcal{S}, \tag{10.20}$$

whereas, based on the mean rate-of-rotation $\bar{\Omega}_{ij}$ Baldwin and Lomax (1978) proposed

$$\nu_T = \ell_m^2 (2\bar{\Omega}_{ij}\bar{\Omega}_{ij})^{\frac{1}{2}} = \ell_m^2 \Omega. \tag{10.21}$$

(Both of these formulae reduce to Eq. 10.19 in the case that $\partial\langle U_1 \rangle/\partial x_2$ is the only non-zero mean velocity gradient.)

In its generalized form, the mixing-length model is applicable to all turbulent flows, and it is arguably the simplest turbulence model. Its major drawback, however, is incompleteness: the mixing length $\ell_m(\mathbf{x})$ has to be specified, and the appropriate specification is inevitably dependent on the geometry of the flow. For a complex flow that has not been studied before, the specification of $\ell_m(\mathbf{x})$ requires a large measure of guesswork, and consequently one should have little confidence in the accuracy of the resulting calculated mean velocity field. On the other hand, there are classes of technologically-important flows which have been studied extensively, so that the appropriate specifications of $\ell_m(\mathbf{x})$ are well established. The prime example is boundary-layer flows in aeronautical applications. The Cebeci-Smith model (Smith and Cebeci 1967) and the Baldwin-Lomax model (Baldwin and Lomax 1978) provide mixing-length specifications that yield quite accurate calculations of attached boundary layers. Details of these models and their performance are provided by Wilcox (1993).

As illustrated in the following exercise, the mixing-length model can also be applied to free shear flows. The predicted mean velocity profile agrees well with experimental data (see, e.g., Schlichting 1979). An interesting (though non-physical) feature of the solution is that the mixing layer has a definite edge at which the mean velocity goes to the free-stream velocity with zero slope but non-zero curvature.

Exercise 10.2 Consider the self-similar temporal mixing layer in which the mean lateral velocity $\langle V \rangle$ is zero, and the axial velocity $\langle U \rangle$ depends only on y and t. The velocity difference is U_s, so that the boundary conditions are $\langle U \rangle = \pm\frac{1}{2}U_s$ at $y = \pm\infty$. The thickness of the layer $\delta(t)$ is defined (as in Fig. 5.21 on page 145) such that $\langle U \rangle = \pm\frac{2}{5}U_s$ at $y = \pm\frac{1}{2}\delta$.

The mixing-length model is applied to this flow, with the mixing length being uniform across the flow and proportional to the flow width, i.e., $\ell_m = \alpha\delta$, where α is a specified constant.

Starting from the Reynolds equations

$$\frac{\partial\langle U \rangle}{\partial t} = -\frac{\partial\langle uv \rangle}{\partial y}, \tag{10.22}$$

show that the mixing length hypothesis implies:

$$\frac{\partial\langle U \rangle}{\partial t} = 2\alpha^2\delta^2\frac{\partial\langle U \rangle}{\partial y}\frac{\partial^2\langle U \rangle}{\partial y^2}. \tag{10.23}$$

Show that this equation admits a self-similar solution of the form $\langle U \rangle = U_s f(\xi)$, where $\xi = y/\delta$; and that $f(\xi)$ satisfied the ordinary differential equation

$$-S\xi f' = 2\alpha^2 f' f'', \tag{10.24}$$

where $S \equiv U_s^{-1}\mathrm{d}\delta/\mathrm{d}t$ is the spreading rate.

Show that Eq. (10.24) admits two different solutions (denoted by f_1 and f_2):

$$f_1 = -\frac{S}{12\alpha^2}\xi^3 + A\xi + B, \tag{10.25}$$

and

$$f_2 = C, \tag{10.26}$$

where A, B and C are arbitrary constants.

The appropriate solution for f is made up of three parts. For $|\xi|$ greater than a particular value ξ^*, f is constant (i.e., f_2):

$$\begin{aligned} f &= -\tfrac{1}{2} \quad \text{for} \quad \xi < -\xi^*, \\ &= \ \ \tfrac{1}{2} \quad \text{for} \quad \xi > \xi^*. \end{aligned} \tag{10.27}$$

Show that the appropriate solution for $-\xi^* < \xi < \xi^*$ satisfying $f'(\pm\xi^*) = 0$ is

$$f = \frac{3}{4}\frac{\xi}{\xi^*} - \frac{1}{4}\left(\frac{\xi}{\xi^*}\right)^3. \tag{10.28}$$

Show that the spreading rate is related to the mixing-length constant by

$$S = 3\alpha^2/{\xi^*}^3, \tag{10.29}$$

and use the definition of δ (i.e., $f\left(\frac{1}{2}\right) = \frac{2}{5}$) to obtain

$$\xi^* \approx 0.8450. \tag{10.30}$$

How does ν_T vary across the flow?

10.3 Turbulent Kinetic Energy Models

With the turbulent viscosity written as

$$\nu_T = \ell^* u^*, \tag{10.31}$$

in the mixing-length model the lengthscale is $\ell^* = \ell_m$ and the velocity scale is (in simple shear flow)

$$u^* = \ell_m \left|\frac{\partial \langle U \rangle}{\partial y}\right|. \tag{10.32}$$

The implication is that the velocity scale is locally determined by the mean velocity gradient; and, in particular, u^* is zero where $\partial\langle U \rangle/\partial y$ is zero. In fact, contrary to this implication, there are several circumstances where the velocity gradient is zero and yet the turbulent velocity scale is non-zero. One example is decaying grid turbulence; another is the center-line of the round jet where direct measurement shows ν_T to be far from zero (see Fig. 5.10 on page 111).

Independently, Kolmogorov (1942) and Prandtl (1945) suggested that it is better to base the velocity scale on the turbulent kinetic energy, i.e.,

$$u^* = ck^{\frac{1}{2}}, \tag{10.33}$$

where c is a constant. If the lengthscale is again taken to be the mixing length, then the turbulent viscosity becomes

$$\nu_T = ck^{\frac{1}{2}}\ell_m. \tag{10.34}$$

As shown in Exercise 10.3, the value of the constant $c \approx 0.55$ yields the correct behavior in the log-law region.

In order for Eq. (10.34) to be used, the value of $k(\mathbf{x}, t)$ must be known or estimated. Kolmogorov and Prandtl suggested achieving this by solving a modelled transport equation for k. This is called a *one-equation model*, because a modelled transport equation is solved for just one turbulence quantity, namely, k.

Before discussing the modelled transport equation for k, it is helpful to itemize all the components of the model:

(i) the mixing length $\ell_m(\mathbf{x}, t)$ is specified

(ii) a modelled transport equation is solved for $k(\mathbf{x}, t)$

(iii) the turbulent viscosity is defined by $\nu_T = c k^{\frac{1}{2}} \ell_m$

(iv) the Reynolds-stresses are obtained from the turbulent viscosity hypothesis, Eq. (10.1)

(v) the Reynolds equations are solved for $\langle \mathbf{U}(\mathbf{x}, t)\rangle$ and $\langle p(\mathbf{x}, t)\rangle$.

Thus, from the specification of ℓ_m and from the solutions to the exact and modelled equations, the following fields are determined: $\langle \mathbf{U}\rangle$, $\langle p\rangle$, ℓ_m, k, ν_T and $\langle u_i u_j\rangle$. These are referred to as "knowns."

We now consider the modelled transport equation for k. The exact equation (Eq. 5.132) is

$$\begin{aligned} \frac{\bar{\mathrm{D}}k}{\bar{\mathrm{D}}t} &\equiv \frac{\partial k}{\partial t} + \langle \mathbf{U}\rangle \cdot \nabla k \\ &= -\nabla \cdot \mathbf{T}' + \mathcal{P} - \varepsilon, \end{aligned} \tag{10.35}$$

where the flux $\mathbf{T}'$ (Eq. 5.140) is

$$T'_i = \tfrac{1}{2}\langle u_i u_j u_j\rangle + \langle u_i p'\rangle/\rho - 2\nu\langle u_j s_{ij}\rangle. \tag{10.36}$$

In Eq. (10.35), any term that is completely determined by the "knowns" is said to be "in closed form." Specifically, $\bar{\mathrm{D}}k/\bar{\mathrm{D}}t$ and $\mathcal{P}$ are in closed form. Conversely, the remaining terms (ε and $\nabla \cdot T'$) are "unknown"; and, in order to obtain a closed set of model equations, these terms must be modelled. That is, "closure approximations" are required that model the unknowns in terms of the knowns.

As discussed extensively in Chapter 6, at high Reynolds number the dissipation rate ε scales as u_0^3/ℓ_0, where u_0 and ℓ_0 are the velocity and length

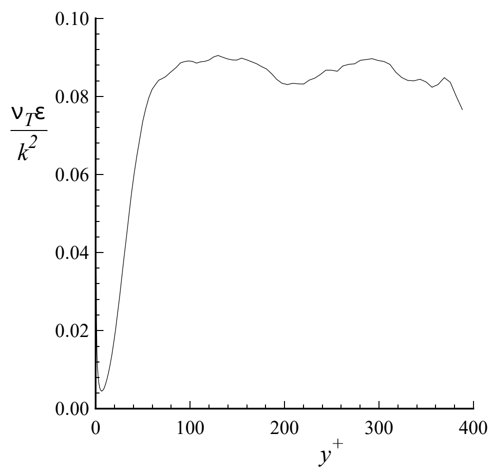

Figure 10.3: Profile of $\nu_T\varepsilon/k^2$ (see Eq. 10.39) from DNS of channel flow at Re $= 13,750$ (Kim et al. 1987).

scales of the energy-containing motions. Consequently, it is reasonable to model ε as

$$\varepsilon = C_D k^{\frac{3}{2}}/\ell_m, \tag{10.37}$$

where C_D is a model constant. Indeed, an examination of the log-law region (Exercise 10.3) yields this relation with $C_D = c^3$.

Modelling assumptions such as Eq. (10.37) deserve close scrutiny. Equations (10.34) and (10.37) can be combined to eliminate ℓ_m to yield

$$\nu_T = cC_D k^2/\varepsilon, \tag{10.38}$$

or, equivalently,

$$\frac{\nu_T\varepsilon}{k^2} = cC_D. \tag{10.39}$$

For simple shear flows, k, ε and $\nu_T = -\langle uv\rangle/(\partial\langle U\rangle/\partial y)$ can be measured, so that this modelling assumption can be tested directly. Figure 10.3 shows the left-hand side of Eq. (10.39) extracted from DNS data of fully-developed turbulent channel flow. It may be seen that (except close to the wall, $y^+ < 50$) this quantity is indeed approximately constant, with a value around 0.09. Figure 10.4 shows the same quantity for the temporal mixing layer: except near the edges, the value is everywhere close to 0.08.

The remaining unknown in the turbulent kinetic energy equation is the energy flux $\mathbf{T}'$ (Eq. 10.36). This is modelled by a gradient diffusion hypothesis as

$$\mathbf{T}' = -\frac{\nu_T}{\sigma_k}\nabla k, \tag{10.40}$$

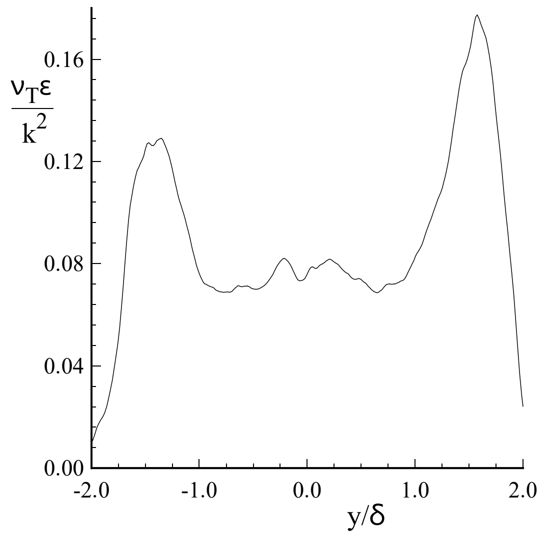

Figure 10.4: Profile of $\nu_T\varepsilon/k^2$ (see Eq. 10.39) from DNS of the temporal mixing layer (from data of Rogers and Moser 1994).

where the "turbulent Prandtl number" for kinetic energy[1] is generally taken to be $\sigma_k = 1.0$. Physically, Eq. (10.40) asserts that (due to velocity and pressure fluctuations) there is a flux of k down the gradient of k. Mathematically, the term ensures that the resulting modelled transport equation for k yields smooth solutions, and that a boundary condition can be imposed on k everywhere on the boundary of the solution domain.

In summary, the one-equation model based on k consists of the modelled transport equation

$$\frac{\bar{\mathrm{D}}k}{\bar{\mathrm{D}}t} = \nabla \cdot \left(\frac{\nu_T}{\sigma_k}\nabla k\right) + \mathcal{P} - \varepsilon, \tag{10.41}$$

with $\nu_T = ck^{\frac{1}{2}}\ell_m$ and $\varepsilon = C_D k^{\frac{3}{2}}/\ell_m$, together with the turbulent viscosity hypothesis (Eq. 10.1) and the specification of ℓ_m.

A comparison of model predictions with experimental data (Wilcox 1993) shows that this one-equation model has a modest advantage in accuracy over mixing-length models. But the major drawback of incompleteness remains: the length scale $\ell_m(\mathbf{x})$ must be specified.

Exercise 10.3 Consider the log-law region of a wall-bounded flow. Use the log-law and the specification $\ell_m = \kappa y$ to show that the appro-

[1]The symbol σ_k is standard notation. Note, however, that σ_k is a scalar, and that "k" is not a suffix in the sense of Cartesian tensor suffix notation.